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FUZZY SET THEORY $\subseteq$ PROBABILITY THEORY?

A COMMENT ON "MEMBERSHIP FUNCTIONS AND PROBABILITY MEASURES OF FUZZY SETS".

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Abstract

From both a theoretical and practical perspective, the integration of fuzzy set theory and probability theory is of particular importance. Both fuzzy set theory and probability theory come with their own formalism for dealing with uncertainty and the question naturally arises whether, and to what extent, these formalisms are different. In this paper we show that formally fuzzy set theory can be formulated in terms of probability theory. This integration is practically significant as it makes results from one world available to researchers in the other world.

Key words: Membership functions; Fuzzy set theory.

1. Introduction

Singpurwalla and Booker (2004) attempt to make *fuzzy set theory* and *probability theory* work in concert. The reason for making such an attempt is the alleged incompatibility of fuzzy set theory and probability theory. In this paper we show that fuzzy set theory can be formulated in terms of probability theory. Specifically, we define the membership function $m_{\tilde{A}}(x)$ for the fuzzy set $\tilde{A}$ as follows:

$$
\begin{aligned}
m_{\tilde{A}}(x) &\stackrel{\text{def}}{=} P((X,Y) \in A | X = x) \qquad (1)\\
&= \int_{A(x)} f(y|X=x)dy \quad \text{with} \quad A(x) = \{y : (x,y) \in A\}
\end{aligned}
$$

It is important to observe that A is a *crisp* set and that x may either be interpreted as a value of the parameter X or as the realization of the random variable X. In general, we use $\tilde{A}$ to denote the fuzzy set corresponding to the crisp set A.

In Section 1 we show that every membership function can be written in this form and that the algebra of membership functions supports a probabilistic interpretation. In Section 2 we show that both the probability measures of Zadeh (1968) and Singpurwalla and Booker (2004) are, distinct, valid probability measures.

According to our definition of the membership function in (1), the membership function can be formulated in terms of probability theory. We see that if we were to observe both x and y it would be possible to classify with respect to the crisp set A. That is, fuzzy sets are the result of a *missing data* situation. The distribution of Y conditionally on $X = x$ characterizes the inference about the missing data y on the basis of the observed data x. Singpurwalla and Booker (2004) reserve a key role for the *Laplacean genie*. In our conception of the membership function (1), the Laplacean genie is *he who knows* y, and hence is able to classify with precision according to the crisp set A.

2. The algebra of membership functions

Zadeh (1965) introduced the following fuzzy set operators:

$$m_{\tilde{A}^c}(x) \stackrel{\text{def}}{=} 1 - m_{\tilde{A}}(x)$$

$$m_{\tilde{A}\cup\tilde{B}}(x) \stackrel{\text{def}}{=} \max(m_{\tilde{A}}(x), m_{\tilde{B}}(x))$$

$$m_{\tilde{A}\cap\tilde{B}}(x) \stackrel{\text{def}}{=} \min(m_{\tilde{A}}(x), m_{\tilde{B}}(x))$$

$$m_{\tilde{A}\bullet\tilde{B}}(x) \stackrel{\text{def}}{=} m_{\tilde{A}}(x)m_{\tilde{B}}(x)$$

$$m_{\tilde{A}+\tilde{B}}(x) \stackrel{\text{def}}{=} m_{\tilde{A}}(x) + m_{\tilde{B}}(x) - m_{\tilde{A}}(x)m_{\tilde{B}}(x)$$

We show that these fuzzy set operations logically follow from the conception of the membership function as a (conditional) probability. Singpurwalla and Booker (2004) observe that they are unable to interpret the product ($\bullet$) and sum (+) of fuzzy sets. It will become clear in the sequel that the interpretation of the product and sum pose no special difficulties from our point of view.

Let y, z, y^c, and z^c be scalar valued variables, and define $A(x)$ and $B(x)$, as well as $A^c(x)$ and $B^c(x)$, as follows, for all x:

$$A(x) = \{y : y \leq m_{\tilde{A}}(x)\} \tag{2}$$

$$B(x) = \{z : z \leq m_{\tilde{B}}(x)\}$$

$$A^c(x) = \{y^c : y^c \leq m_{\tilde{A}^c}(x)\}$$

$$B^c(x) = \{z^c : z^c \leq m_{\tilde{B}^c}(x)\}$$

Furthermore, assume that

$$(Y|X = x) \sim (Z|X = x) \sim (Y^c|X = x) \sim (Z^c|X = x) \sim \mathcal{U}(0, 1) \quad . \tag{3}$$

Notice that y^c is not necessarily some form of complement, such as $1 - y$, of y. In the following we discuss their relation in more detail. Under these

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conditions, it holds that

$$m_{\tilde{A}}(x) = P(Y \in A(x)|X = x) = P((X,Y) \in A|X = x)$$

$$m_{\tilde{B}}(x) = P(Z \in B(x)|X = x) = P((X,Z) \in B|X = x)$$

$$m_{\tilde{A}^c}(x) = P(Y^c \in A^c(x)|X = x) = P((X,Y^c) \in A^c|X = x) = 1 - m_{\tilde{A}}(x)$$

$$m_{\tilde{B}^c}(x) = P(Z^c \in B^c(x)|X = x) = P((X,Z^c) \in B^c|X = x) = 1 - m_{\tilde{B}}(x) \quad .$$

The union (or) and intersection (and) operations satisfy the following relationship:

$$\begin{aligned} P((X,Y) \in A|X = x) + P((X,Z) \in B|X = x) \\ &= P((X,Y) \in A \text{ and } (X,Z) \in B|X = x) \\ &+ P((X,Y) \in A \text{ or } (X,Z) \in B|X = x) \end{aligned}$$

which can be expressed in the notation of fuzzy set theory as follows:

$$m_{\tilde{A}}(x) + m_{\tilde{B}}(x) = m_{\tilde{A} \text{ and } \tilde{B}}(x) + m_{\tilde{A} \text{ or } \tilde{B}}(x)$$

It is readily observed that this relationship is satisfied both by the pair $(\cap, \cup)$ and by the pair $(\bullet, +)$. That is, both $+$ and $\cup$ are valid union operators, and both $\bullet$ and $\cap$ are valid intersection operators. To us, this reflects that fuzzy set theory is not conceptually rich enough to uniquely define its concepts. It will become clear that this lack of uniqueness is the reason why the union and sum, and for that matter, the intersection and product operators of Zadeh (1965) are difficult to interpret.

From our point of view, the problem with defining the union and intersection is that we have considerable freedom in specifying the joint distribution for Y and Z conditionally on $X = x$ subject to the constraint in (3). It is well known that the joint distribution of Z and Y conditionally on $X = x$, obeys the following constraints:

$$\max(0, P(Y \leq y|X = x) + P(Z \leq z|X = x) - 1)$$
$$\leq P(Y \leq y, Z \leq z|X = x) \leq$$
$$\min(P(Y \leq y|X = x), P(Z \leq z|X = x))$$

where the lower and upper boundary distributions are the Fréchet distributions corresponding to maximal negative and positive dependence, respectively. In the present setting maximal negative dependence corresponds to setting Y equal to $1-Z$, whereas maximal positive dependence corresponds to setting Y equal to Z. If we let Y be equal to Z we obtain the intersection operator $\cap$ of Zadeh (1965) for the intersection

$$P((X, Y) \in A \text{ and } (X, Z) \in B|X = x)$$
$$= \min(P((X, Y) \in A|X = x), P((X, Z) \in B|X = x))$$

and the union operator $\cup$ for the union:

$$P((X,Y) \in A \text{ or } (X,Z) \in B | X = x)$$

$$= \max(P((X,Y) \in A | X = x), P((X,Z) \in B | X = x))$$

If we let Y and Z be independent we obtain the product operator $\bullet$ of Zadeh (1965) for the intersection

$$P((X,Y) \in A \text{ and } (X,Z) \in B | X = x)$$

$$= P((X,Y) \in A | X = x) P((X,Z) \in B | X = x)$$

and the sum operator $+$ for the union:

$$P((X,Y) \in A \text{ or } (X,Z) \in B | X = x)$$

$$= P((X,Y) \in A | X = x) + P((X,Z) \in B | X = x)$$

$$- P((X,Y) \in A | X = x) P((X,Z) \in B | X = x)$$

Finally, if we let Y be equal to $1 - Z$ we obtain the bounded difference $\Cap$

for the intersection

$$P((X,Y)\in A \text{ and } (X,Z)\in B|X=x)$$
$$= \max(0, P((X,Y)\in A|X=x) - (1 - P((X,Z)\in B|X=x)))$$

and the bounded sum $\Cup$ for the union

$$P((X,Y)\in A \text{ or } (X,Z)\in B|X=x)$$
$$= \min(1, P((X,Y)\in A|X=x) + P((X,Z)\in B|X=x))$$

It is clear that the same reasoning applies when A^c is considered rather then B. For instance, with Y equal to Y^c the intersection is not empty but can be expressed as follows:

$$P((X,Y)\in A \text{ and } (X,Y^c)\in A^c|X=x)$$
$$= \min(P((X,Y)\in A|X=x), P((X,Y^c)\in A^c|X=x))$$
$$= \min(P((X,Y)\in A|X=x), 1 - P((X,Y)\in A|X=x))$$

Summarizing, we have shown that for any pair of membership functions we can specify crisp sets A, B, A^c, and B^c, and a distribution for Y, Z, Y^c, and Z^c conditionally on $X = x$ such that the normal algebra of fuzzy sets holds. We have also shown that the sum and product operators of Zadeh (1965) are nothing but the union and intersection operators for a particular joint distribution for Y and Z conditionally on $X = x$.

3. Probability Measures Associated with Fuzzy Sets

Zadeh (1968) proposed the following probability measure for a fuzzy set $\tilde{A}$

$$\begin{aligned} \Pi(\tilde{A}) &= \mathcal{E}(m_{\tilde{A}}(X)) \\ &= \int P((X,Y) \in A | X = x) P(X = x) dx \\ &= P((X,Y) \in A) \end{aligned} \tag{4}$$

It is seen that from our point of view $\Pi(\tilde{A})$ is nothing but the probability measure associated with the crisp set A. Singpurwalla and Booker (2004) claim that $\Pi(\tilde{A})$ is *not* a valid probability measure. However, their objections stem from the lack of conceptual power of fuzzy set theory alluded to in the previous section. Specifically, they claim that the fact that $\Pi(\tilde{A})$ together with $\Pi(\tilde{B})$ are sufficient to determine $\Pi(\tilde{A} \cap \tilde{B})$ contradicts probability theory. However under maximum positive or negative dependence as well as under independence the marginals uniquely determine the joint distribution. Furthermore, they claim that $\Pi(\tilde{A}+\tilde{B})$ has no analog in probability theory. We have found however that the sum operator has an analog in probability theory, and hence so does $\Pi(\tilde{A}+\tilde{B})$. Notice that because the intersection operator is not uniquely defined, the same holds for conditional probability and the notion of independence.

Singpurwalla and Booker (2004) propose another probability measure for the fuzzy set $\tilde{A}$. Adopting our notation and viewpoint, their measure can be written in a slightly more general form as follows:

$$
\begin{aligned}
&P((X,Y)\in A|(X,Z)\in B)\\
&\qquad = \sum_x P((X,Y)\in A|X=x)P(X=x|(X,Z)\in B)\\
&\qquad = \sum_x P((X,Y)\in A|X=x)\frac{P((X,Z)\in B|X=x)P(X=x)}{\sum_x P((X,Z)\in B|X=x)P(X=x)}\\
&\qquad = \frac{\sum_x P((X,Y)\in A|X=x)P((X,Z)\in B|X=x)P(X=x)}{\sum_x P((X,Z)\in B|X=x)P(X=x)}\\
&\qquad \propto \sum_x P((X,Y)\in A|X=x)P((X,Z)\in B|X=x)P(X=x)
\end{aligned}
$$

It is important to observe that in this derivation we tacitly assumed that Y and Z are independent conditionally on $X=x$. Singpurwalla and Booker (2004) consider the situation where $A=B$ and interpret it as the probability of the fuzzy set $\tilde{A}$ after having been supplied with the membership function $m_{\tilde{A}}(x)$ of an expert.

The only relevant difference between the probability measures proposed by Zadeh (1968) and Singpurwalla and Booker (2004) is that the former considers the marginal probability of the set $(X,Y) \in A$, whereas the latter consider this probability conditionally on the event $(X,Z) \in B$ under the additional assumption that Y and Z are independent conditionally on $X = x$.

From the viewpoint developed in this paper we can also address other probability measures. For instance, given my membership function for the fuzzy set $\tilde{B}$ we get the updated fuzzy set probability measure for the set $\tilde{A}$:

$$P((X,Y) \in A|(X,Y) \in B)$$
$$= \sum_x \min\left(\frac{P((X,Y) \in A|X = x)}{P((X,Y) \in B|X = x)}, 1\right) P(X = x|(X,Y) \in B)$$

In writing the last equation we have assumed that *the same* Y is used both for classifying in the set B as for classifying in the set A.

4. Discussion

We have shown that fuzzy set theory can be formulated in terms of probability theory. Here we put our formal results in perspective. It is important to observe that probability theory is treated from a purely formal point of view as part of mathematical measure theory. That is, the usual parlance of betting and random experiments is not necessarily invoked by probability theory. The question we tried to address in this paper is whether fuzzy set theory is a different formal system than probability theory. We found that, in fact, the answer to this question is no.

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